

# Critical Number of Flavours in QED

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We demonstrate that in unquenched quantum electrodynamics (QED), chiral symmetry breaking ceases to exist above a critical number of fermion flavours  $N_f$ . This is a necessary and sufficient consequence of the fact that there exists a critical value of electromagnetic coupling  $\alpha$  beyond which dynamical mass generation gets triggered. We employ a multiplicatively renormalizable photon propagator involving leading logarithms to all orders in  $\alpha$  to illustrate this. We study the flavour and coupling dependence of the dynamically generated mass analytically as well as numerically. We also derive the scaling laws for the dynamical mass as a function of  $\alpha$  and  $N_f$ . Up to a multiplicative constant, these scaling laws are related through  $(\alpha, \alpha_c) \leftrightarrow (1/N_f, 1/N_f^c)$ . Calculation of the mass anomalous dimension  $\gamma_m$  shows that it is always greater than its value in the quenched case. We also evaluate the  $\beta$ -function. The criticality plane is drawn in the  $(\alpha, N_f)$  phase space which clearly depicts how larger  $N_f$  is required to restore chiral symmetry for an increasing interaction strength.

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The  $\beta$ -function determines the running of the coupling constant. In quantum chromodynamics (QCD), its evolution, both in the ultraviolet and infrared, is crucially influenced by the number of light quark flavours. Virtual quarks and gluons contribute to its perturbative tail in a diametrically opposed manner and the value of  $N_f$  determines which would be the dominant effect. QCD exhibits asymptotic freedom because  $N_f$  happens to be less than a critical value of  $N_f^{c1} = 16.5$ . Lattice studies in the infrared indicate that just below this value, chiral symmetry remains unbroken and colour degrees of freedom are unconfined [1]. Below this conformal window, for an  $8 < N_f^{c2} < \sim 12$ , the evolution of the beta function in the infrared is such that QCD enters the phase of dynamical mass generation (DMG) as well as confinement. QCD is not the only gauge theory where infrared dynamics responds to the number of fermion flavours in such a dramatic fashion. It has been established that QED3 possess a critical number of flavours  $N_f^{c3}$  associated with the simultaneous emergence of dynamical masses and confinement if the electron wave function renormalization, photon vacuum polarization and electron photon vertex are homogeneous functions at the infrared momenta, [2, 5]. See [3] and [4] for some original works on DMG in QCD and QED3, respectively, through the Schwinger-Dyson equations (SDEs).

One ponders if such criticality also characterizes other gauge theories in a similar manner. In this paper, we study the flavour dependence of DMG in QED4 or QED, [6]. For large  $\alpha$ , it is known to exhibit chiral symmetry breaking in the one loop approximation of the photon propagator, [7]. A consistent solution for coupled equations for the fermion mass function and photon wave

function renormalization in the bare vertex approximation was obtained in [8]. In this article, we demonstrate that the unquenched QED also has a critical number of flavours  $N_f^c$  above which chiral symmetry is restored. The starting point is the SDE for the electron propagator

$$S^{-1}(p) = S_0^{-1}(p) + ie^2 \int d^4k \gamma^\mu S(k) \Gamma^\nu(k, p) \Delta_{\mu\nu}(q), \quad (1)$$

where  $q = k - p$ ,  $e$  is the electromagnetic coupling and  $S_0^{-1}(p) = \not{p}$  is the inverse bare propagator for massless electrons. We parameterize the full propagator  $S(p)$  in terms of the electron wave function renormalization  $F(p^2)$  and the mass function  $M(p^2)$  as  $S(p) = F(p^2)/(\not{p} - M(p^2))$ .  $\Delta_{\mu\nu}(q)$  is the full photon propagator which can be conveniently written as

$$\Delta_{\mu\nu}(q) = -\frac{G(q^2)}{q^2} \left( g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) - \xi \frac{q_\mu q_\nu}{q^4}, \quad (2)$$

where  $\xi$  is the covariant gauge parameter such that  $\xi = 0$  corresponds to the Landau gauge.  $G(q^2)$  is the photon renormalization function. The full electron photon vertex is represented by  $\Gamma^\mu(k, p)$ . The form of the full vertex is tightly constrained by various key properties of the gauge theory, [9], e.g., multiplicative renormalizability of the fermion and the gauge boson propagators, [10, 11], perturbation theory, [12], the requirements of gauge invariance/covariance, [13–17] and of course, observed phenomenology, [18]. The most general decomposition of this vertex in terms of its longitudinal and transverse components is

$$\Gamma^\mu(k, p) = \sum_{i=1}^4 \lambda_i(k, p) L_i^\mu(k, p) + \sum_{i=1}^8 \tau_i(k, p) T_i^\mu(k, p), \quad (3)$$

$N_f$	$\alpha_c^A$	$\alpha_c^N$	$\alpha$	$N_f^{cA}$	$N_f^{cN}$
<b>0.5</b>	1.69	1.7405	<b>2.5</b>	1.18	1.0253
<b>1.0</b>	2.27	2.4590	<b>3.0</b>	1.54	1.3205
<b>1.5</b>	2.94	3.3056	<b>3.5</b>	1.86	1.6209
<b>2.0</b>	3.74	3.9879	<b>4.0</b>	2.15	2.0123

TABLE I: We tabulate analytical (indicated with superscript A) and numerical values (indicated with superscript N) of  $\alpha_c$  for different values of  $N_f$ , and  $N_f^c$  for different values of  $\alpha$ .

where  $L_1^\mu = \gamma^\mu$ ,  $L_2^\mu = (k+p)^\mu (\not{k} + \not{p})$ ,  $L_3^\mu = (k+p)^\mu$  and  $L_4^\mu = \sigma^{\mu\nu} (k+p)_\nu$ , where  $\sigma^{\mu\nu} = [\gamma^\mu, \gamma^\nu]/2$ . The coefficients  $\lambda_i$  are determined through the Ward-Takahashi identity relating the electron propagator with the electron photon vertex [23] :

$$\begin{aligned}\lambda_1(k, p) &= \frac{1}{2} \left[ \frac{1}{F(k^2)} + \frac{1}{F(p^2)} \right], \\ \lambda_2(k, p) &= \frac{1}{2} \frac{1}{k^2 - p^2} \left[ \frac{1}{F(k^2)} - \frac{1}{F(p^2)} \right], \\ \lambda_3(k, p) &= -\frac{1}{k^2 - p^2} \left[ \frac{M(k^2)}{F(k^2)} - \frac{M(p^2)}{F(p^2)} \right]\end{aligned}\quad (4)$$

and  $\lambda_4(k, p) = 0$ . A simple choice of the transverse coefficients which, combined with the longitudinal component, renders both  $F(p^2)$  the  $G(q^2)$  multiplicatively renormalizable for massless fermions, has been constructed only recently [11]. The longitudinal coefficient  $\lambda_1$  plays a crucial role in ensuring the correct leading logarithms are summed up for the photon wave function renormalization. Similarly  $\lambda_2$  dictates the multiplicative renormalizability of the photon propagator. Using this information, the *ansatz* proposed in [11] makes use of the following four transverse basis vectors as suggested by Ball and Chiu, [23] :

$$\begin{aligned}T_2^\mu(k, p) &= p^\mu k \cdot q - k^\mu p \cdot q, \\ T_3^\mu(k, p) &= q^2 \gamma^\mu - q^\mu \not{q}, \\ T_6^\mu(k, p) &= -\gamma^\mu (k^2 - p^2) + (k+p)^\mu \not{q}, \\ T_8^\mu(k, p) &= -\gamma^\mu k^\lambda p^\nu \sigma_{\lambda\nu} + k^\mu \not{p} - p^\mu \not{k}.\end{aligned}\quad (5)$$

The corresponding coefficients are chosen to depend upon  $F(p^2)$  in the following simple manner :

$$\begin{aligned}\tau_2 &= \frac{-4}{3(k^4 - p^4)} \left[ \frac{1}{F(k^2)} - \frac{1}{F(p^2)} \right] - \frac{2}{3(k^2 + p^2)^2} \\ &\times \left[ \frac{1}{F(k^2)} + \frac{1}{F(p^2)} \right] \ln \left[ \frac{1}{2} \left( \frac{F(q^2)}{F(k^2)} + \frac{F(q^2)}{F(p^2)} \right) \right], \\ \tau_3 &= \frac{5}{12(k^2 - p^2)} \left[ \frac{1}{F(k^2)} - \frac{1}{F(p^2)} \right] + \frac{1}{3(k^2 + p^2)} \\ &\times \left[ \frac{1}{F(k^2)} + \frac{1}{F(p^2)} \right] \ln \left[ \frac{1}{2} \left( \frac{F(q^2)}{F(k^2)} + \frac{F(q^2)}{F(p^2)} \right) \right], \\ \tau_6 &= \frac{-1}{4(k^2 + p^2)} \left[ \frac{1}{F(k^2)} - \frac{1}{F(p^2)} \right], \\ \tau_8 &= 0.\end{aligned}\quad (6)$$

The coefficient  $\lambda_3$ , which enters into the description of massive fermions, is irrelevant to the power law behaviour of both  $F(p^2)$  and  $G(q^2)$ . However, it is intimately related to the value of the anomalous dimension  $\gamma_m$  for the fermion mass function. In the quenched theory, the ultraviolet behaviour of  $M(p^2)$  can be expressed as

$$M(p^2) \sim (p^2)^{\gamma_m/2-1} \quad (7)$$

in the deep Euclidean region. At criticality, the mass function behaves as Eq. (7) at all momenta. If the transverse vertex vanishes in the Landau gauge,  $\gamma_m = 1.058$ , see e.g. [19]. However, Holdom and Mahanta [20], using the arguments based on the Cornwall-Jackiw-Tomboulis (CJT) effective potential technique, have shown that  $\gamma_m$  is strictly equal to 1. The importance and usefulness of employing the bare vertex was also stressed in [21]. If it were true that  $\gamma_m = 1$ , this would suggest that there is a necessary piece in the transverse part of the effective vertex which does not vanish in the Landau gauge. Complete calculation of the fermion-boson vertex at the one loop in arbitrary gauge and dimensions, [22], reveals that the transverse part of the vertex indeed does not vanish in the Landau gauge in any space-time dimensions. This fact may possibly favour Holdom's arguments. Thus it may well be that the non-zero transverse piece in the Landau gauge cancels out the  $\lambda_3$  piece of the longitudinal component in the equation for the mass function. Considering this argument, one such vertex was constructed in [15]. Following suit, consider the following full vertex

$$\Gamma^\mu(k, p) = \Gamma_{BC}^\mu(k, p) + \Gamma_{KP}^\mu(k, p) + \Gamma_A^\mu(k, p). \quad (8)$$

As the subscripts indicate,  $\Gamma_{BC}^\mu(k, p)$  is the longitudinal Ball-Chiu vertex, defined by Eq. (4),  $\Gamma_{KP}^\mu(k, p)$  is the proposal by Kizilersu and Pennington, Eq. (6), and  $\Gamma_A^\mu(k, p)$  is the additional transverse piece which minimally ensures  $\gamma_m = 1$  in the quenched case. With this choice of the full vertex, we obtain, in the massless limit

$$F(p^2) = \left( \frac{p^2}{\Lambda^2} \right)^\nu \quad G(q^2) = \left( \frac{q^2}{\Lambda^2} \right)^s, \quad (9)$$

where  $\nu = \alpha\xi/(4\pi)$ ,  $s = \alpha N_f/(3\pi)$ ,  $\alpha = e^2/(4\pi)$  and  $N_f$  is the number of massless fermion flavours. All main conclusions are robust under different truncations, e.g., for 1-loop logarithmic photon propagator and for a resummation of the propagators beyond leading logs. Near criticality, where the generated masses are small, one can assume that the power law solutions for the propagators capture the correct description of chiral symmetry breaking. We choose to study the resulting equation for the mass function in the convenient Landau gauge. Results for any other gauge can be derived by applying the Landau-Khalatnikov-Fradkin transformations [5, 17, 24]. The usual simplifying assumption  $G(q^2) = G(k^2)$  for  $k^2 > p^2$  and  $G(q^2) = G(p^2)$  for  $p^2 > k^2$  allows the analytical treatment of the linearized equation for the mass function :

$$M(p^2) = \frac{g(p^2)}{p^2} \int_{m^2}^{p^2} dk^2 M(k^2) + \int_{p^2}^{\Lambda^2} dk^2 \frac{M(k^2)}{k^2} g(k^2), \quad (10)$$

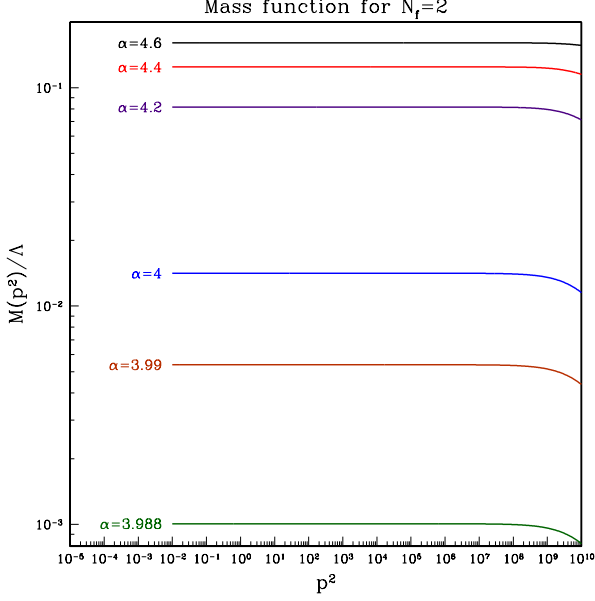


FIG. 1: The mass function for different values of  $\alpha$ .

where  $g(q^2) = 3\alpha G(q^2)/(4\pi)$  and we have introduced ultraviolet cut-off  $\Lambda^2$ . The infrared cut-off  $m^2$  mimics the  $M^2(k^2)$  term in the denominator which has been dropped off. It is already known that for the one loop photon propagator, there exists a critical coupling  $\alpha_c$  above which masses are dynamically generated. One can formally demonstrate that the existence of critical coupling implies the existence of a critical number of flavours above which chiral symmetry is restored. Note that in Eq. (10),  $G(q^2) \equiv G(q^2, \alpha N_f)$ . Instead of working with the variables  $(\alpha, N_f)$ , if we define  $\alpha' = \alpha N_f$ , we could equally work with  $(\alpha', N_f)$ . In such case,  $g(q^2) = 3\alpha' G(q^2)/(4N_f\pi)$  with  $G(q^2) \equiv G(q^2, \alpha')$ . If we hold  $\alpha'$  constant, the effective coupling is  $1/N_f$ . Therefore, the presence of an  $\alpha_c$  implies the existence of an  $N_f^c$ . The critical behaviour should thus translate as  $(\alpha, \alpha_c) \rightarrow (1/N_f, 1/N_f^c)$ . Moreover, if there is no critical  $\alpha$ , there will be no critical  $N_f$ . We shall demonstrate this explicitly for the multiplicatively renormalizable photon propagator. Let us make the change of variables  $x = \Lambda^2/p^2$  and convert the integral equation (10) into a second order differential equation

$$x^2 M''(x) + sx M'(x) + \frac{3\alpha}{4\pi}(1-s) \frac{M(x)}{x^s} = 0 \quad (11)$$

with the following infrared and ultraviolet boundary conditions respectively

$$M(1) = M'(1)/(1-s), \quad M'(\Lambda^2/m^2) = 0. \quad (12)$$

Eq. (11) can be converted into a Bessel equation through Lommel transformations :  $z = Bx^\gamma$ ,  $W = x^{-\lambda} M$ . Thus we work with  $W(z)$  instead of  $M(x)$ . The corresponding equation is

$$z^2 W''(z) + z W'(z) + (z^2 - A^2) W(z) = 0, \quad (13)$$

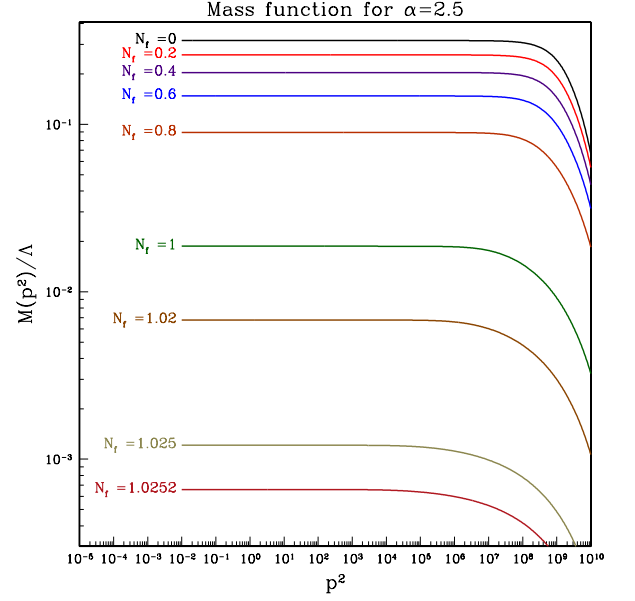


FIG. 2: The mass function for a fixed  $\alpha$  and varying  $N_f$ . As  $N_f$  is reduced, the mass function drops significantly for increasingly small variations in  $N_f$ , suggesting the existence of a critical number of flavours.

where  $\gamma = -s/2$ ,  $\lambda = (1-s)/2$ ,  $A = (1-s)/s$  and  $B = \sqrt{3\alpha(1-s)/(\pi s^2)}$ . Moreover, we have assumed  $s < 1$ . Eq. (13) has the solution

$$W(z) = C_1 J_A(z) + C_2 Y_A(z), \quad (14)$$

where  $J_A(z)$  and  $Y_A(z)$  are the Bessel functions of the first and the second kind respectively. The boundary conditions get translated as

$$W'(z) = \frac{1-s-\lambda}{\gamma B} W(z) \Big|_{z=B},$$

$$\alpha W(z) + \gamma B \left( \frac{\Lambda^2}{m^2} \right)^\gamma W'(z) = 0 \Big|_{z=B \left( \frac{\Lambda^2}{m^2} \right)^\gamma}. \quad (15)$$

These conditions allow us to find the constants  $C_1$  and  $C_2$ , and the equation for the mass  $m$  :

$$\frac{2\lambda J_A(z) + \gamma B (\Lambda^2/m^2)^\gamma [J_{A-1}(z) - J_{A+1}(z)]}{2\lambda Y_A(z) + \gamma B (\Lambda^2/m^2)^\gamma [Y_{A-1}(z) - Y_{A+1}(z)]} \Big|_{z=B \left( \frac{\Lambda^2}{m^2} \right)^\gamma}$$

$$= \frac{\gamma B [J_{A-1}(B) - J_{A+1}(B)] - 2(1-s-\lambda)J_A(B)}{\gamma B [Y_{A-1}(B) - Y_{A+1}(B)] - 2(1-s-\lambda)Y_A(B)}. \quad (16)$$

Critical  $\alpha$  or  $N_f$  can be obtained from this equation by requiring it to hold true for  $\Lambda \rightarrow \infty$ . This implies finding the zeros of the equation

$$\gamma B [J_{A-1}(B) - J_{A+1}(B)] - 2(1-s-\lambda)J_A(B) = 0. \quad (17)$$

For various values of  $N_f$ , analytical values of  $\alpha_c$  have been tabulated in the second column of Table. I. Similarly, for

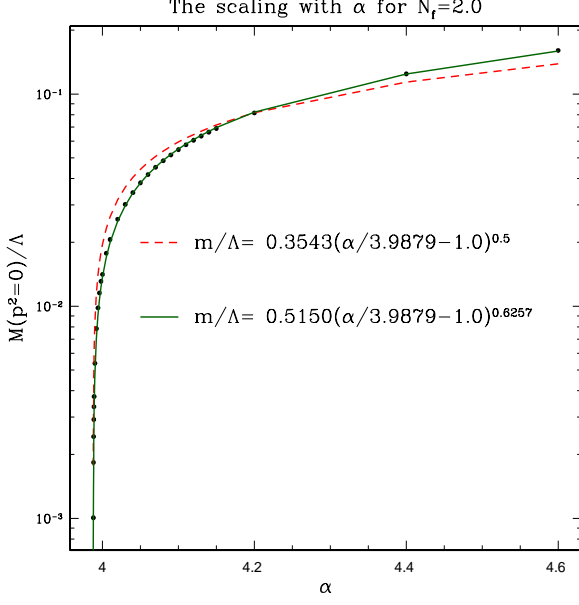


FIG. 3: The scaling law for the coupling  $\alpha$ . Numerical solution (solid line) is compared with analytical prediction (dashed line) of the linearized equation.

$N_f$	$1/b_\alpha$	$\alpha$	$1/b_{N_f}$
0.5	1.0313	2.5	0.7629
1.0	0.7819	3.0	0.6944
1.5	0.6882	3.5	0.7314
2.0	0.6257	4.0	0.7640

TABLE II: The numerical results for the scaling law.

various  $\alpha$ ,  $N_f^c$  has been tabulated in the fifth column of the same table. In order to arrive at the scaling laws, we expand the left hand side of Eq. (16) in powers of  $m^2/\Lambda^2$  and keep the leading terms. Thus

$$\frac{m^2}{\Lambda^2} \equiv f(\alpha, N_f) = \left(\frac{B}{2}\right)^{-2/s} \Gamma(A) \Gamma(A+2) \frac{(\alpha - \gamma A)}{2\pi\gamma} \times \frac{\gamma B [J_{A-1}(B) - J_{A+1}(B)] - 2(1-s-\lambda)J_A(B)}{\gamma B [Y_{A-1}(B) - Y_{A+1}(B)] - 2(1-s-\lambda)Y_A(B)}. \quad (18)$$

Carrying out a Taylor expansion near the critical coupling, we find the following scaling law

$$m/\Lambda = h_\alpha(N_f) (\alpha - \alpha_c)^{1/2}, \quad (19)$$

where  $h_\alpha(N_f) = \sqrt{\partial f / \partial \alpha|_{\alpha=\alpha_c}}$ . As anticipated, the scaling law for  $N_f$  comes out to be of the form :

$$m/\Lambda = h_{N_f}(\alpha) (N_f^c - N_f)^{1/2}, \quad (20)$$

with  $h_{N_f}(\alpha) = \sqrt{\partial f / \partial N_f|_{N_f=N_f^c}}$ . These analytical results are based upon the linearization of the original problem and the identification of  $M(p^2 \rightarrow 0) = m$ . Exact numerical analysis of the original non-linearized version of

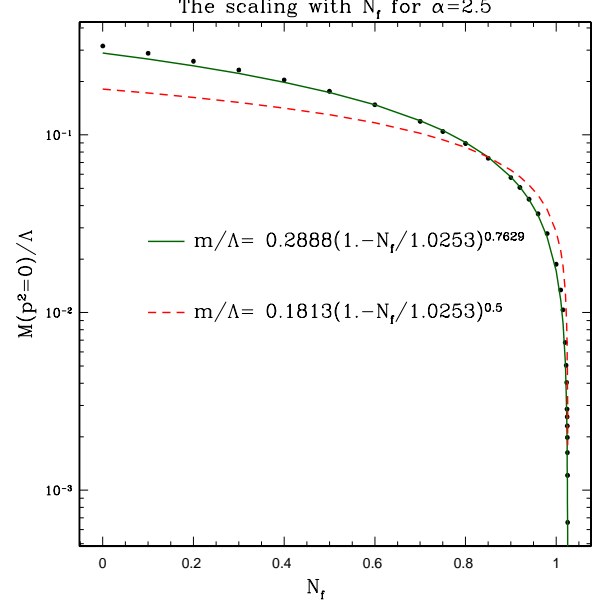


FIG. 4: The scaling law for the dynamically generated mass as a function of  $N_f$ . The solid line is the fit to the numerical findings whereas the dashed line is the analytical result of the linearized equation for the mass function.

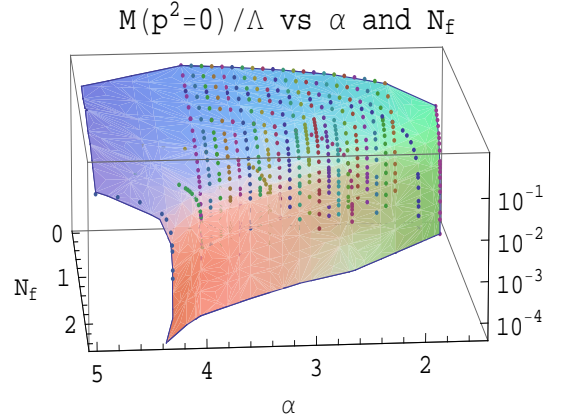


FIG. 5: The criticality plane for the dynamical mass in the phase space of  $\alpha$  and  $N_f$ . The points shown are the numerical results obtained.

Eq. (10) confirms the qualitative nature of the aforementioned analytical results. In Figs. (1) and (2), we depict the mass functions for different values of  $\alpha$  and  $N_f$ . A study of the dependence of  $M(0) \equiv m$  as a function of  $\alpha$  and  $N_f$ , exemplified in Figs. (3) and (4), permits us to decipher the corresponding critical values and scaling laws. The numerical details differ slightly from the analytical findings. We now have the scaling laws

$$m/\Lambda = a_\alpha(N_f)(\alpha - \alpha_c)^{1/b_\alpha(N_f)}, \quad (21)$$

$$m/\Lambda = a_{N_f}(\alpha)(N_f^c - N_f)^{1/b_{N_f}(\alpha)}. \quad (22)$$

Though none of  $b_\alpha(N_f)$  and  $b_{N_f}(\alpha)$  is strictly 2 for a

broader fit of the scaling law (see Table II), figures (3) and (4) show that the analytical results are not a bad representation of the exact results in the immediate vicinity of the critical coupling (compare the solid curves against the dashed ones). The infinite order phase transition of the quenched QED softens out to a finite order transition in its unquenched version. It also has consequences for the mass anomalous dimensions. If  $B \approx C \equiv |A^2 - 1/4|$ , the large momentum behaviour of Eq. (14) implies  $\gamma_m \approx 1 + s$ . On the other hand, if  $B \gg C$ ,  $\gamma_m \approx 1 + s/2$ . The quenched limit trivially follows. The numerical analysis of the full equation also yields  $\gamma_m > 1$ . In order to obtain a finite electron mass in the limit of  $\Lambda \rightarrow \infty$ , one requires charge renormalization [26]. Therefore, in this limit, we impose

$$\alpha(\Lambda) = \alpha_c + \frac{1}{(a_\alpha(N_f))^{b_\alpha(N_f)}} \left[ \frac{m}{\Lambda} \right]^{b_\alpha(N_f)}. \quad (23)$$

It implies the following  $\beta$ -function

$$\beta(\alpha) = \Lambda \partial\alpha/\partial\Lambda = -b_\alpha(N_f) (\alpha - \alpha_c). \quad (24)$$

Thus the  $\beta$ -function has a stable zero at the point  $\alpha = \alpha_c$ , the result also obtained with the one loop approximation to the photon propagator, [7]. Not only does a critical value of coupling separate chirally asymmetric and symmetric phases but so does also a critical number of flavours above which fermions cease to possess mass just like in QCD and QED3. Fig. (5) shows the criticality plane in the phase space of  $\alpha$  and  $N_f$ . It interpolates and extrapolates the points obtained through the numerical analysis. We believe that our analysis can and should be extended to the study of QCD through its SDEs. This is for the future.

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